

Harvesting Logistic Populations

In this lab, we will consider the effects of populations of fish that grow logistically and are harvested for food. We will see that for a given population, there is *critical* harvesting rate that determines the long-term sustainability of the population. This is called the *Maximum Sustainable Yield*.

Recall the **logistic equation**

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L}\right), P(0) = P_0$$

Its solution is the **the logistic function**:

$$P(t) = \frac{L}{1 + Ae^{-kt}} \quad \text{with} \quad A = \frac{L - P_0}{P_0}.$$

Review

Refer back to Lab 9 (Questions 6-9), and to Worksheet 13-3 for this section.

- (a) Draw a graph of $\frac{dP}{dt}$ vs. P for the logistic equation. Note that this is *not* the graph of the solution. That's P vs t !
(b) Label the roots of your graph. What feature of the differential equation do your roots correspond to?
(c) What term do we use for the constant L in terms of the population?
(d) For what value of P does $\frac{dP}{dt}$ reach a maximum? What is $\frac{dP}{dt}$ at this population? Add this point to your graph and label it with its coordinates. What feature of the graph of the solution $P(t)$ does this maximum correspond to?

Using the Applet

In this week's lab, we will use a custom-written¹ applet to understand the logistic equation. Open the Geogebra applet. The applet displays the fish population of a fishery (in thousands of fish) over time (in years), modeled by the logistic equation with $k = 1 \text{ year}^{-1}$ and $L = 100$ thousand fish.

You will see two graphs:

- The slopefield for $\frac{dP}{dt} = P \left(1 - \frac{P}{100}\right)$ with two solution curves plotted:
 - $P(0) = 1$ thousand fish
 - $P(0) = 110$ thousand fish
- The graph of $\frac{dP}{dt}$ vs. P .

Note that the roots of the graph of $\frac{dP}{dt}$ vs. P correspond precisely to the two equilibria of the differential equation: a population of zero fish, and the carrying capacity, 100 thousand fish.

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Harvesting

Up to now, we have considered an undisturbed population modeled by a logistic equation. In this lab, we consider the effect of harvesting on the population. By *harvesting*, we mean removing a *fixed* number of fish from the population per unit time. Note that this is distinct from removing a *proportion* of the population. We deal with the latter case in class and focus on the former here.

2. First, a thought experiment:
 - (a) Suppose first that the fish population is large (near its carrying capacity). If a relatively small number of fish were harvested each year, what effect would you expect to see on the fish population?
 - (b) Next, suppose the fish population is very small. What would harvesting fish do to the population?
3. Now back to the math: suppose that a constant number H of fish per year are removed from the population. The model is now

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L} \right) - H.$$

Use the slider in the applet to increase H from zero fish per year to $H = 20$ thousand fish per year to see a model of what happens when 20 thousand fish are harvested per year.

- (a) Note the shift in the graph of $\frac{dP}{dt}$ vs. P down by 20 units. What is the effect of this on the roots/equilibria?
- (b) The applet now displays three solution curves: one with $P(0) = 110$ thousand fish, one with $P(0)$ just above the lower equilibrium, and another with $P(0)$ just below it. By examining these curves, determine the stability of each of the equilibria.
- (c) Now reconsider your thought experiment from the previous question:
 - i. Suppose that the fishery initially has a fish population *higher than the lower equilibrium*. What will happen over time? Why does it make sense that the long-term population of fish is no longer the carrying capacity of 100 thousand fish, but is lower instead?
 - ii. Suppose instead that the fish population is initially *less than the lower equilibrium*. What will happen over time? Explain why this happens.

Sustainable Harvesting

We saw in the previous part that if we harvest a small number of fish, and the initial population is sufficiently high, the fish population is stable over time. We now turn to quantifying just how many fish can be harvested to maintain this stability, and understanding the consequences if this limit is exceeded.

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4. (a) Consider the general logistic model $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$. What is the maximal value of $\frac{dP}{dt}$ in term of k and L ? (Refer back to question 1(d).) For reasons we will see shortly, we call this value the *maximal sustainable yield* (MSY) of the fishery, and denote it by H_{MSY} .
- (b) In the case displayed on the applet, what is the exact value of H_{MSY} ? Locate it on the graph of $\frac{dP}{dt}$ vs. P . Use the slider to display the model when $H = H_{\text{MSY}}$, then answer these questions:
- Suppose we harvest exactly $H = F_{\text{MSY}}$ fish per year. Why would we then only have one equilibrium? Would it be stable or unstable?
 - If $H = F_{\text{MSY}}$, it is possible for the fishery to be viable over time? Under what circumstances?
5. (a) In theory, with a sufficiently large population, harvesting exactly H_{MSY} fish per year appears to be sustainable, as our model does not predict a collapse of the fishery. Why is this a bad idea in practice? Why might harvesting exactly the MSY of fish lead to collapse in practice?
- (b) Understanding this, the local government proposes to allow harvesting of just a small amount less than the MSY of fish. Move the slider to show the situation when $H = 24.9$ thousand fish per year. Observing the graphs, give arguments for and against this policy. Consider that there may be varying environmental conditions in the real world!

Overfishing and Collapse

Suppose that the harvesting rate is greater than H_{MSY} . We refer to this as *overfishing*.

6. Move the slider to show the situation when we harvest $H = 27$ thousand fish per year.
- Are there any equilibria? What about the graph of P vs. $\frac{dP}{dt}$ tells you that solutions still have an inflection point?
 - Describe in words the change in fish population under conditions of such overfishing. Carefully refer to changing rates of population decrease in your answer: are the rates of collapse always accelerating? Explain.

Even in conditions of overfishing, it may be possible to intervene to prevent total collapse of a fishery. However, since H_{MSY} is often very hard to compute in practice (as the values of k and L themselves are not easily discoverable), this can be difficult.

7. For this question, you will need to examine how the solution curves behave *as H moves* from a little below H_{MSY} (say, $H = 24$ thousand fish per years) to above (say, $H = 27$ thousand fish per year).
- By considering the upper solution at time $t = 8$ years as H moves from 24 to 27 thousand fish per year, explain why it might be very difficult to detect whether a population is being overfished or not merely by knowing the fish population level and its trend, without knowledge of the value of H_{MSY} .
 - Explain why, in the case of overfishing, once the population is below the inflection point, the situation is dire and critical, and action should be taken immediately.

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(c) Suppose that the local government decides to intervene to mandate lower fishing levels, as it suspects overfishing.

- i. Suppose that at the time of intervention, the fish population is still above the IP. Explain why, at least in theory, if fishing is reduced to *any amount* below H_{MSY} , the fish population is likely to recover.

(Hint: The IP is always at $P = \frac{L}{2}$. In our case, this is 50 thousand fish. What happens to the fish population if, say $P = 60$ thousand fish and H is reduced to be any amount below $H_{MSY} = 25$ thousand fish per year?)

- ii. Suppose by contrast that the fish population is below the IP. Explain why in this case, the *amount below H_{MSY} to which fishing is reduced* may determine whether or not the population will recover or still die out.

(Hint: in our case, suppose that 35 thousand fish remain. Compare what happens if H is reduced to 24 thousand fish per year vs. if H is reduced to 20 thousand fish per year.)

- iii. The government argues that since fish stocks appear to be dropping, it is necessary to temporarily completely stop fishing to ensure the recovery and long-term sustainability of the population. The union of workers who rely on fishing for a living argues that such drastic measures are not necessary. Give mathematical justifications for both arguments. Keep in mind that parameters are hard to measure (k , L , and the current population in particular).

Report

Hand in answers to Questions 3(c), 5, and 7. Be sure to write in complete sentences, explain your work carefully, and include any plots from the applet as needed.