Newton's Law of Motion

Purpose: The purpose of this lab is to introduce you to differential equations. As an important application, we will study Newton's Law of Motion and help you understand why Newton's calculations caused a revolution in human thought.

Part I: Introduction to Differential Equations

> A **differential equation** is an equation involving the derivatives of an unknown function.
>
> Finding the unknown function is called **solving** the differential equation.

The following example shows that we may be able to solve differential equations by reverse engineering our knowledge of derivatives.

Example 1:

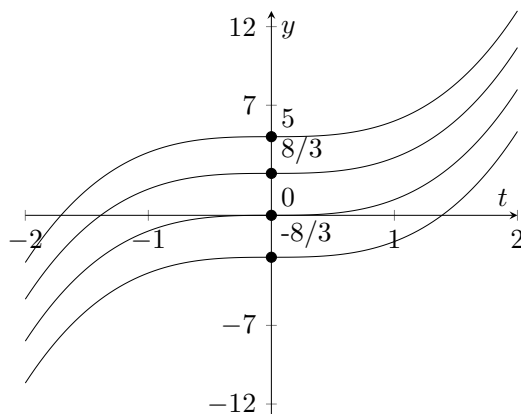Suppose that an unknown function $y(t)$ satisfies

$$y'(t) = t^2.$$

By differentiating, it is easy to see that the function $t^3/3$ satisfies the differential equation. However, this is not the only function which satisfies $y'(t) = t^2$, since for any constant c, the function

$$y(t) = \frac{t^3}{3} + c$$

also satisfies the differential equation. Thus, the differential equation has infinitely many solutions, each one corresponding to a different value of c. Now, suppose we know the value of y at $t = 0$. For example, suppose

$$y(0) = 5.$$

Then there is only one function in our family of functions, namely $y(t) = t^3/3 + 5$, which satisfies both the differential equation and the "initial condition" $y(0) = 5$. Several different solutions of the differential equation corresponding to different initial conditions are shown below.

In Example 1, $t^3/3$ is called an **antiderivative** of t^2. Thus, the procedure that we used to solve $y'(t) = t^2$ was first to find an antiderivative of t^2. Then we observed that for each choice of c, the function $t^3/3 + c$ was also a solution. For instance, in the last example, $c = 5$.

Example 2:

We want a function $y(x)$ satisfying $y'(x) = 3x^2 + 2$ and $y(0) = 4$. By trial and error we find that $x^3 + 2x$ is an antiderivative of $3x^2 + 2$. So every function of the form

$$y(x) = x^3 + 2x + c$$

satisfies the differential equation. Since $4 = y(0) = 0^3 + 0 + c$, we see that $c = 4$. Thus, the solution of the differential equation which satisfies $y(0) = 4$ is

$$y(x) = x^3 + 2x + 4.$$

A differential equation together with an initial condition on the solution is called an
initial value problem.

---

1. Find all antiderivatives of each of the following functions.

    (a) $2t + 3$

    (b) $t^2 + t^7$

    (c) $t^2 - 7t + 5$

2. Solve the following initial value problems.

    (a) $y'(t) = 2t + 3$, $y(0) = 4$

    (b) $y'(t) = 2t + 3$, $y(1) = 8$ (Careful!)

    (c) $y'(t) = t^2 + t^7$, $y(0) = 3$

    (d) $y'(t) = -\sin t$, $y(0) = 1$

    (e) $y'(t) = e^t$, $y(\ln 2) = 0$.

The **order** of a differential equation is the highest derivative that occurs in the equation.

106L Labs: Newton's Law of Motion

In Examples 1 and 2 we solved first order differential equations. In the following example we shall solve a second order differential equation.

Example 3:

We wish to solve the differential equation $s''(t) = t^2$ with initial conditions $s(0) = 2$ and $s'(0) = 4$.

Because solving the differential equation means finding the function $s(t)$ and because $s''(t)$ is the *second* derivative of $s(t)$, we will have to antidifferentiate twice. An antiderivative for t^2 is $t^3/3$. Thus, as we saw earlier, all functions of the form $t^3/3 + b$ are antiderivatives, where b is a constant:

$$s'(t) = \frac{t^3}{3} + b, \tag{1}$$

then $s''(t) = t^2$. To see that this is true, simply differentiate both sides of (1).

We need to antidifferentiate again to find $s(t)$: a family of antiderivatives for $t^3/3 + b$ is $t^4/12 + bt + c$ for any constant c. So

$$s(t) = \frac{1}{12}t^4 + bt + c.$$

You should check this by explicit differentiation.

We shall now use the initial conditions to determine b and c. Substituting $t = 0$ into $s(t) = 1/12t^4 + bt + c$, we find $2 = s(0) = 0 + 0 + c$. Therefore, $c = 2$.

Substituting $t = 0$ into $s'(t) = t^3/3 + b$, we find $4 = s'(0) = 0 + b$. Therefore, $b = 4$.

So the function

$$s(t) = \frac{t^4}{12} + 4t + 2$$

satisfies both the differential equation and the initial conditions. You should also check this directly.

From this example we can see a strategy for solving differential equations of the form

$$s''(t) = f(t):$$

- Antidifferentiate $s''(t)$ to find $s'(t)$. Be sure you add a constant, say b!

  - If a value of $s'(t)$ is known, find a value for b.

- Antidifferentiate $s'(t)$ to find $s(t)$. Be sure you add a constant, say c!

  - If a value of $s(t)$ is known, find a value for c.

3. Solve the following initial value problems.

  (a) $y''(t) = 3t + 2$, $y(0) = 4, y'(0) = 1$

  (b) $y''(t) = t^2 - 5$, $y(1) = -2, y'(1) = 2$

  (c) $y''(t) = \cos(t)$, $y(\pi) = 2$, $y'(\pi) = 1$

4. Write short clear descriptions of the following terms.

  (a) differential equation

  (b) antiderivative

  (c) initial value problem

  (d) order of a differential equation

Plug in!

You will learn several different methods for solving initial value problems. Whatever the method, you can always check that the "solution" really is a solution by plugging it into the differential equation and checking whether the differential equation is satisfied. Similarly, you can always check that the "solution" satisfies the initial conditions by evaluating the solution at the initial point.

Part II: Newton's Law of Motion

Newton's Law of Motion, $F = ma$, expresses a relationship among the force F on an object, the mass m of the object, and the acceleration a of the object. We shall discuss Newton's Law of Motion for several special cases in which the object moves along a line. If the function $s(t)$ gives the position of the object on the line at time t, then the rate of change of $s(t)$, namely $s'(t)$, is called the velocity, and the rate of change of $s'(t)$, namely $s''(t)$, is called the acceleration.

Newton's Law of Motion tells us about the second derivative of $s(t)$. We use this information about the second derivative and information about the object at time $t = 0$ to calculate $s(t)$ itself.

Falling Bodies

Suppose that a rock is dropped from a height of 200 feet. We calculate its position as a function of time as it falls to the ground. We measure time t in seconds starting at the time of release and distance in feet. We denote the height above the ground at time t by the function $h(t)$. The "line" is the vertical line between the initial position of the rock and the place where it hits the ground.

$$
\begin{aligned}
h(t) &= \text{distance above the ground} \\
h'(t) &= \text{rate of change of distance } = \text{ velocity} \\
h''(t) &= \text{rate of change of velocity } = \text{ acceleration}
\end{aligned}
$$

We are given the information that $h(0) = 200$ feet.

We assume that the only force acting on the rock is the force due to gravity. For a rock this assumption is reasonable; for a feather it would not be reasonable since the resistance of the air would significantly affect the feather. Near the surface of the earth the force of gravity is

$$F = -mg$$

where $g = 32$ feet per second per second. So, in this case, Newton's Law of Motion can be written

$$mh''(t) = -m(32). \tag{2}$$

$h''(t)$ is negative since the force pushes in the direction that causes h to decrease. Dividing by m:

$$h''(t) = -32 \text{ ft/sec}^2.$$

This is a second order differential equation, we expect that we will need two initial conditions to determine the solution. We were told explicitly that

$$h(0) = 200 \text{ ft.}$$

The other condition is implicit in the description of the problem we are trying to solve. We were told that the rock "is dropped," which says that it was not thrown down or thrown up, but released with initial velocity zero. That is,

$$h'(0) = 0 \text{ ft/sec.}$$

5. Solve the initial value problem for the differential equation and show that $h(t) = -16t^2 + 200$.

Now that we have computed $h(t)$, the height of the rock at all times t (until it hits the earth), you can use $h(t)$ to calculate other quantities of interest. For example, the height after 2 seconds is $h(2) = (-16)2^2 + 200 = 136$ feet. The distance fallen in t seconds is

$$h(0) - h(t) = 200 - (-16t^2 + 200) = 16t^2 \text{ feet.}$$

6. For what values of t will $h(t)$ actually represent the height of the rock above the ground?

7. Suppose that the rock discussed above is not dropped but instead is thrown upward (from a platform 200 feet above the ground) with an initial velocity of 100 feet per second.

   (a) Let $h(t)$ be the distance above the ground as a function of time. Write down the differential equation and initial conditions that are to be satisfied by $h(t)$.

   (b) Compute $h(t)$ and use it to determine the height above ground at times $t = 1, 2, \ldots, 7$.

   (c) Compute the time when the rock will hit the ground.

   (d) Compute the greatest height above the ground. (*Hint:* This will happen at the time when the velocity is zero.)

Notice that the mass m appears on both sides of Newton's equation of motion for falling bodies and therefore cancels out. Thus, the motion is independent of the mass, a fact discovered by Galileo. This is strictly true only in a vacuum since the resistance of air exerts a force counter to any motion and the resistance depends on the size and surface properties of the body. The motion *does* depend on g, which is 32 ft/s^2 on the surface of the earth. If we had asked the same questions on the surface of the moon, where $g = 5$ ft/s^2, the answers would have been different.

Motion on a Line

Suppose that an object of mass m moves along a line. The object could be a ping-pong ball or a subatomic particle. Think of the line as the x-axis and let $s(t)$ denote the position of the object on the axis at time t. The time t is measured in some time units (seconds, minutes, or hours) and s is measured in some distance units (feet or meters). If the object is pushed by a constant force F acting parallel to the line, then, according to Newton's Law of Motion,

$$s''(t) = \frac{F}{m}$$

so the acceleration is constant. The appropriate units for F depend on the units chosen for s and t. For simplicity, we let all the units remain unspecified. If $F > 0$, the force acts to push the object in the positive x-direction. Similarly, if $s'(t) > 0$, then the object is moving in the positive x-direction.

8. Suppose that $F/m = 5$ and that the particle starts two units to the right of the origin (that is, $s(0) = 2$) with initial velocity $s'(0) = -10$. Describe the subsequent motion of the particle. (Hint: First use the differential equation and the initial conditions to find $s(t)$. Then describe the motion using that fact that $s(t)$ is the position and $s'(t)$ is the velocity at each time t.)

9. In elementary textbooks it is often stated that "Newton's first law says that objects will move with constant velocity if the force applied to them is zero." Explain why this follows from Newton's Law of Motion. (Hint: if the force is zero, what can you say about acceleration?)

10. A particle moves on a line with acceleration $s''(t) = -4t$. Suppose that $s(0) = 6$ and $s'(0) = 24$. Find the functions $s(t)$ and $s'(t)$. Graph the functions $s(t)$, $s'(t)$, and $s''(t)$ on the same set of axes and describe the motion in words.

Report - Understanding Gravity

In the lab, it was stated that near the surface of the Earth, the force acting on an object was $F = -mg$, where g is the Earth's acceleration due to gravity. This report will explore why that is true and apply the ideas involved to computations involving other planets and celestial objects.

1. **Universal Gravitation**

   Newton's Law of Universal Gravitation states that the force F between two objects is directly proportional to the product of their masses, m_1 and m_2, and inversely proportional to the square of the distance between them, r.

   (a) Write this as an equation involving F, m_1, m_2, r, and a constant G. (See question 9 from Worksheet 1-3 of 105L.)

   (b) Do some research online to find the value and units of G, assuming the masses are measured in kilograms and the distance in meters. Be sure to cite your source.

2. **Gravity on Earth**

   When the law of universal gravitation is applied to celestial objects like planets, we consider the distance between two objects to be the distance between their centers. That is, the distance between your desk and the Earth for these purposes is the radius of the Earth. When we say that an object is 'near the surface' of the Earth, we mean that we can neglect the distance between the Earth's surface and the object for the purpose of gravity computations, as that distance is much much smaller than the Earth's radius.

   For our computations, the 'distance between the Earth and the object' is simply the radius of the Earth.

   (a) Find the radius of the Earth (in meters) and its mass (in kilograms) online. Cite your source.

   (b) Use the law of universal gravitation to explain why, for an object with mass m near the surface of the Earth, the planet exerts a force of $F = -mg$ on it. Use part (a) to compute g to one decimal place. Include units. (Hint: Use Question 1, with r being the radius of the Earth, and m_2 being its mass.)

3. **Acceleration Due to Gravity Elsewhere**

   For each of the following celestial objects, find its radius and mass online (cite sources). Then compute the acceleration due to gravity on each. Rather than repeat the same computation over and over, set up a spreadsheet with columns 'Name', 'Mass', 'Radius', and 'g'. Insert data in the first three columns, then use a formula to compute the fourth. Do this for Mercury, Venus, Earth, Mars, Ganymede, the Moon, Europa, Titan, Jupiter, and the Sun.

   When you hand this report in, copy your spreadsheet into a table. Also, show the formula you used to compute the fourth column.

7

4. **Experimental Computations**

In the lab, we showed that the motion of an object under gravity can be modeled using the second order differential equation $h''(t) = -g$, where h is the height of the object at time t. If an object is thrown from height h_0 above the surface a planet with initial velocity v_0, we have initial conditions $h(0) = h_0$ and $h'(0) = v_0$. We can use this to design an experiment to compute g and other data about a planet.

(a) Suppose that an object is thrown upward from the surface of a planet with initial velocity 25 meters per second. Write down and solve an initial value problem to show that its height at time t seconds is $h(t) = -\frac{1}{2}gt^2 + 25t$ meters. (Hint: this is very similar to Question 1 in Part II of the lab.)

(b) Suppose that the object takes 20 seconds to land back on the surface. Compute g for the planet. (Hint: use your answer to part (a), recalling that if the object is on the surface, its height is zero meters!)

(c) The planet was photographed from space and found to have a radius of 10,000km. Use the law of universal gravitation and your answer to part (b) to compute its mass. Be careful with units! How many times more massive is this planet than Earth?