Sequences, Sums, and Sigma (Σ) Notation

Sequences

Definition A *sequence* is an ordered set of numbers defined by some rule.

Examples

1. Write out completely the sequences given by the following rules:

   (a) $a_i = i^2$, where $0 \le i \le 5$.

   (b) $c_k = \frac{1}{k}$, where $5 \le k < 9$.

   (c) $p_j = 3$, where $3 \le j \le 6$.

   (d) $b_n = (n-3)^2$, where $3 \le n \le 8$ (compare this to question (a) above.)

2. Write a rule for sequences with the given bounds that are identical to the ones above:

   (a) $10 \le i \le 15$ identical to the sequence in (1)(a).

   (b) $1 \le k < 5$ identical to the sequence in (1)(b).

   (c) $-2 \le j \le 1$ identical to the sequence in (1)(c).

   (d) $100 \le n \le 105$ identical to the sequence in (1)(d).

3. Write down rules for the following sequences:

   (a) $1, 8, 27, 64, 125$, starting with index $i = 1$.

   (b) $1, 2, 4, 8, 16, 32$, starting with $j = 1$.

   (c) $1, 2, 4, 8, 16, 32$, starting with $j = 0$.

   (d) $6, 9, 12, 15$, starting with $k = 0$.

   (e) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, starting with $i = 1$.

   (f) $2, 4, \ldots, 10, 12$, starting with $k = 0$.

   (g) $3, 6, \ldots, 102, 105$, starting with $k = 1$.

4. In your own words, explain what the $\ldots$ notation in the last two questions means.

Series

Definition A *series* is the sum of a sequence. We will develop short-hand notation for series, called *Sigma*-notation, after the Greek letter Σ.

Example Consider the sequence $a_i = i^2$, where $0 \leq i \leq 5$ from Question 1 (a). Suppose we want to add it up. We could write

$$0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2,$$

but this is pretty tedious, and will only get more so if our series has, say, 500 terms in it! Instead, we can write the following:

$$\sum_{i=0}^{5} i^2.$$

Note that this contains many features:

- The index (i)
- The index bounds (0 to 5).
- The rule for generating the sequence.
- The Σ indicating that we add the terms up.

Questions

5. For each of the sequences in questions 1, 2, and 3 above, write the corresponding series in Σ-notation.

6. Write out the following in long-form, either completely or using ... notation. Also, write down how many terms are in each of the series:

   (a) $\displaystyle\sum_{i=1}^{5} \frac{5}{i^3}$.

   (b) $\displaystyle\sum_{k=0}^{101} e^{\sin(k)}$.

   (c) $\displaystyle\sum_{m=4}^{9} \frac{5}{(m-3)^3}$ (Compare to (a)!)

   (d) $\displaystyle\sum_{n=6}^{107} e^{\sin(n-6)}$ (Compare to (b)!)

   (e) $\displaystyle\sum_{i=0}^{10} 1$ (For this one, also find the actual sum.)

7. **Reindexing:** Rewrite each of the following sums with the given lower or upper bound, and write how many terms are in each sum. For a hint, see Questions **6(c)** and **6(d)**.

   (a) $\displaystyle\sum_{i=1}^{5} i^2$, with lower bound 0.

   (b) $\displaystyle\sum_{k=1}^{100} \frac{1}{k+1}$, with upper bound 99.

   (c) $\displaystyle\sum_{m=203}^{300} \frac{6}{e^m}$, with lower bound 1.

   (d) $\displaystyle\sum_{m=203}^{300} \frac{6}{e^m}$, with lower bound 0.

Toward Areas Under Curves

8. Write rules for the following sequences with the given bounds:

   (a) $1, 1.6, 2.2, 2.8, 3.4$, with lower bound 0.

   (b) $1.6, 2.2, 2.8, 3.4, 4$, with upper bound 5.

   (c) $\frac{1}{1}, \frac{1}{1.6}, \frac{1}{2.2}, \frac{1}{2.8}, \frac{1}{3.4}$, with lower bound 0.

   (d) $\frac{1}{1.6}, \frac{1}{2.2}, \frac{1}{2.8}, \frac{1}{3.4}, \frac{1}{4}$, with upper bound 5.

9. Write the following series in Σ-notation with the given bounds:

   (a) $\frac{1}{1} \times 0.6 + \frac{1}{1.6} \times 0.6 + \frac{1}{2.2} \times 0.6 + \frac{1}{2.8} \times 0.6 + \frac{1}{3.4} \times 0.6$, with lower bound 0.

   (b) $\frac{1}{1.6} \times 0.6 + \frac{1}{2.2} \times 0.6 + \frac{1}{2.8} \times 0.6 + \frac{1}{3.4} \times 0.6 + \frac{1}{4} \times 0.6$, with upper bound 5.

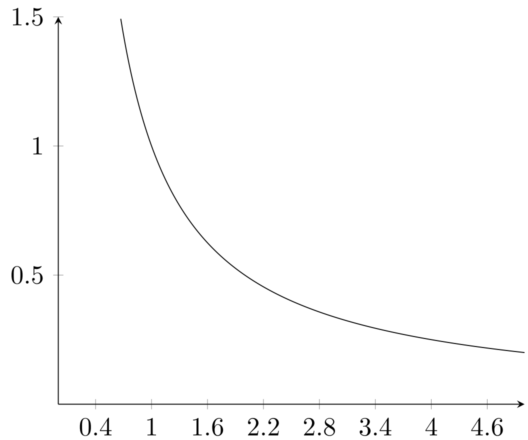
   (c) $\frac{1}{1} \times 0.3 + \frac{1}{1.3} \times 0.3 + \ldots + \frac{1}{3.4} \times 0.3 + \frac{1}{3.7} \times 0.3$, with lower bound 0.

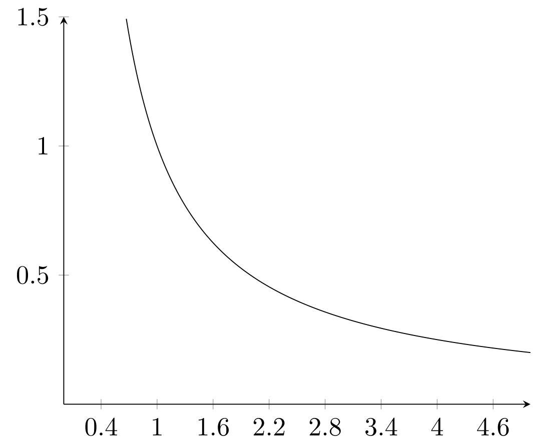
   (d) $\frac{1}{1.3} \times 0.3 + \frac{1}{1.6} \times 0.3 + \ldots + \frac{1}{3.7} \times 0.3 + \frac{1}{4} \times 0.3$, with upper bound bound 10.

10. Consider the curve $f(x) = \frac{1}{x}$. Using methods developed in class, we can estimate the area under this curve from, say, $x = 1$ to $x = 4$. Suppose we do by dividing the interval $[1, 4]$ into $n = 5$ equal subintervals.

    (a) What is the length, Δx, of each subinterval?

(b) On two copies of the curve, draw the rectangles for a left-hand sum (LHS) estimate of the area. On the other, draw the rectangles for a right-hand sum (RHS) estimate of the area.





(c) This part of this question concerns only the LHS:

i. Write down all the elements of the sequence of the left-hand sides of your subintervals.

ii. Write down all the elements of the sequence of the heights of each of your rectangles.

iii. Write down (in long form) the series giving your estimated area under the curve.

(d) This part of this question concerns only the RHS:

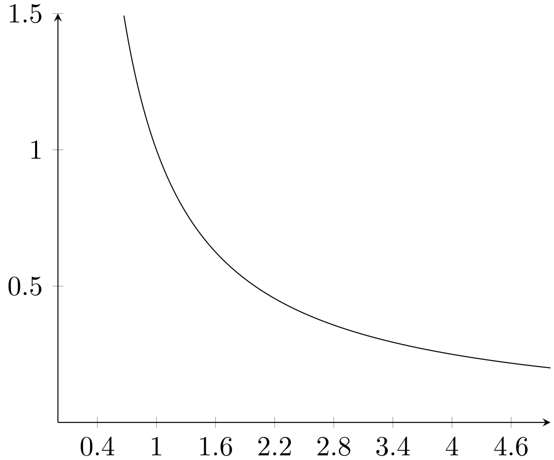
i. Write down all the elements of the sequence of the right-hand sides of your subintervals.

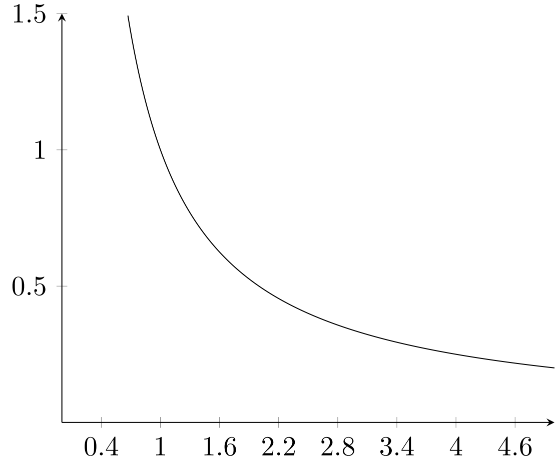
ii. Write down all the elements of the sequence of the heights of each of your rectangles.

iii. Write down (in long form) the series giving your estimated area under the curve.

(e) Write the LHS and RHS in Σ-notation. Compare your answers to questions 9(a) and 9(b) above.

11. Do the previous question again, but with $n = 10$ this time. Compare your final answers to questions 9(c) and 9(d) above.





12. Now suppose $n = 100$. At that point, it seems more than just tedious to write out all the terms. This is where the power of Σ-notation shines! Write down sums for the LHS and RHS for the area under the curve $f(x) = \frac{1}{x}$ between $x = 1$ and $x = 4$ with $n = 100$ subintervals. Be sure to start by figuring out Δx, and perhaps the first (and last) few subintervals.