Playing with Differential Equations

In this lab, we will use a Geogebra based applet that plots slopefields and solutions to explore some differential equations and the behavior of their solutions. To start with, open the applet. You can input a differential equation in the input box, and adjust various parameters. Spend a minute or two:

- playing with the sliders to get a feel for what they do;

- Moving the intial value A around by dragging it to see how the solution curves change;

- Adding more solution curves by checking the boxes. You can move their initial values around indepedently to see more than one solution curve at a time.

The lab will use the applet to explore solution curves to various differential equations with various initial values. As you work through the lab, it will be useful to keep record of these. You can save an image by clicking on the menu icon in the top right of the applet and selecting Export Image. You may need to click on File in the menu first.

Family I: $\frac{dy}{dx} = f(x)$

Consider the differential equation $\frac{dy}{dx} = x^2$.

1. Input x^2 into the applet and look at solutions. Move the initial value to the y-axis, then move it up and down the axis. What do you observe about the different solution curves you get?

2. (a) Solve the differential equation with each of the following initial values (by hand):
    - $y(0) = 2$;
    - $y(0) = -1$;
    - $y(3) = 1$;
    - $y(3) = -1$

   (b) Graphically, how do the first and second solutions compare? What about the third and fourth?

3. By reference to the previous question, make a general statement about differential equations of the form $\frac{dy}{dx} = f(x)$. Explain. (See Questions 3 and 4 on Worksheet 10-2.)

Family II: $\frac{dy}{dx} = k(y - c)$

On the slope fields worksheet, we have been looking at $\frac{dy}{dx} = 2(y - 1)$ and $\frac{dy}{dx} = -2(y - 1)$. Plot slopefields for both of these in two separate windows using the applet.

4. (a) Comment on differences and similarities between the two.

   (b) For each, plot three solution curves: $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$.

(c) For each of the two differential equations, fill in the blanks:

- If $y(0) = 1$, the solution is $y(t) =$ _____. That is, the solution is _____.
- If $y(0) > 1$, solutions are _____ (increasing/decreasing) and _____ (concave up/concave down). As $t \to \infty$, they _____ (approach/move away from) the solution from the previous bullet point.
- If $y(0) < 1$, solutions are _____ (increasing/decreasing) and _____ (concave up/concave down). As $t \to \infty$, they _____ (approach/move away from) the solution from the first bullet point.

5. Plot some more slopefields for differential equations of the form $\frac{dy}{dx} = k(y-c)$ (in the previous question, $k = 2$ or $k = -2$ and $c = 1$). Then answer the following questions:

(a) What happens if $y(0) = c$? These are called *equilibrium* solutions. You will explore them further in class.

(b) What is the biggest difference between positive and negative values of k? Specifically, what happens to solutions as $x \to \infty$ in each case? This has to do with *stability* of the equilibrium.

(c) Plot two slopefields with the same value of c, but different positive values of k. Fill in the blank: Increasing the value of k makes the solutions _____.

k and c are called *parameters* of the *family* of differential equations $\frac{dy}{dx} = k(y - c)$. Understanding how changing parameters changes the behavior of solutions is a very important part of understanding differential equations and their applications.

This particular family of differential equations will play a role in our first population model (the *exponential* model of population growth), as well as in mixing problems, and in Newton's Law of Cooling.

Family III: $\frac{dy}{dx} = ky(L - y)$

Use the applet to plot a slopefield for the differential equation $\frac{dy}{dx} = 0.5y(3 - y)$. Add in solution curves for $y(0) = -1$, $y(0) = 1$, and $y(0) = 5$.

6. What do the solution curves with $y(0) = 0$ and $y(0) = 3$ look like? Try to figure this out by looking at the slopefield without plotting them, then test your hypothesis.

7. In this case, $\frac{dy}{dx}$ depends solely on y, with no dependence on x. In such cases, we can understand some of the behavior of solutions by plotting a graph of $\frac{dy}{dx}$ as a function of y.

(a) Fill in the blanks/cross out the wrong answer: $0.5y(3 - y)$ is a polynomial with degree _____. Its top coefficient is positive/negative. Its roots are $y =$ _____.

(b) Draw a graph of $\frac{dy}{dx}$ vs. y. Be sure to label your axes! By observing your graph, fill in the blanks:

- When $y < 0$, $\frac{dy}{dx}$ is _____. Therefore, any solution curve with $y(0) < 0$ is _____.

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- When $0 < y < 3$, $\frac{dy}{dx}$ is _____. Therefore, any solution curve with $0 < y(0) < 3$ is _____.
- When $y > 3$, $\frac{dy}{dx}$ is _____. Therefore, any solution curve with $y(0) > 3$ is _____.

Make sure you can see how this lines up with the solution curves you see on the slopefield! Understanding the connection between the graph you drew and the various solution curves is very important.

8. How can you detect an inflection point of a function $y(x)$ using a graph of its derivative $\frac{dy}{dx}$? Refer back to the Derivatives and Roots lab from 105L if you don't remember.

    (a) Fill in the blanks: A function $y(x)$ has an inflection point when $\frac{dy}{dx}$ has an _____.

    (b) Referring back to your graph from the previous question, at what exact value of y does a solution curve have an inflection point? Look back at your slopefield and the solution curves and make sure you see this. At this point, the value of $\frac{dy}{dx}$ is _____.

9. Now consider the general case: $\frac{dy}{dx} = ky(L - y)$, with $k, L > 0$. By doing a similar analysis to the one above, including drawing a well labeled graph of $\frac{dy}{dx}$ vs y, answer the following questions. Your answers will be in terms of k, L, or both.

    (a) The differential equation has constant solutions (a.k.a. equilibrium solutions) when $y(0) =$ _____ and when $y(0) =$ _____.

    (b) For which values of $y(0)$ are the solutions curves increasing? For which value of $y(0)$ are they decreasing?

    (c) If a solution curve is increasing, it has an inflection point. At what value of y does the inflection point occur? What is the value of $\frac{dy}{dx}$ at that point?

Note that just as in the previous section, the value of k controls the *steepness* of the solutions: a larger value of k makes the solutions grow faster. L is a *limiting value*. This family of differential equations is called the *logistic* equation. It will be our second model of population growth. It also has widespread applications in economics and other fields.

Report: $\frac{dy}{dx} = x + y$

Plot the slopefield for this equation using the applet. Then add in solutions with initial values $y(0) = 2$, $y(0) = -1$ and $y(0) = -2$. Move these points around a bit to get a feel for the different solutions to this differential equation.

1. First, consider the solution with $y(0) = -1$. Can you find an equation for this solution? Show that your answer is indeed a solution of the differential equation.

2. Considering other solution curves, answer the following:

   (a) Fill in the blanks: If a solution curve lies above the solution found in Question 1, it is _____. If it lies below the solution found in Question 1, it is _____.

   (b) Which of the solution curves have critical points? Are they minima, maxima, or neither?

   (c) What happens to all the solutions as $x \to -\infty$? What do they approach? (Note: I'm not asking if they approach a limit here. Do they approach a *line*? Which one?)

The following questions explain your observations above.

3. By differentiating both sides of the differential equation $\frac{dy}{dx} = x + y$, show that $\frac{d^2y}{dx^2} = 1 + x + y$. Then answer the questions below.

   (a) Suppose that $y = -x - 1$. Plug this into the formula you just showed. What do you conclude? Compare Question 1.

   (b) What can you say about $\frac{d^2y}{dx^2}$ if $y < -x - 1$? What about if $y > -x - 1$? Compare Question **2(a)**.

4. What is $\frac{dy}{dx}$ if $y = -x$?

   (a) Fill in the blank: We can conclude that any solution that passes through the line $y = -x$ has a _____ point there.

   (b) The line $y = -x$ lies above the line $y = -x - 1$. By noting this, and using the conclusion from Question **3(b)**, explain your answer to Question **2(b)**.

5. Show that any function of the form $y(x) = Ce^x - x - 1$ is a solution of the differential equation. Use this to explain your observation in **2(c)**.

When handing in this report, include a picture of the slopefield and relevant solution curves. You can use the menu button in the applet and select 'Export Image'.