Review

1.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+(i-1)\Delta x)\Delta x(\text{Left Hand Sum})$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x(\text{Right Hand Sum})$$

where $\Delta x =$ ____________.

2. Suppose $f(x)$ is decreasing and continuous. Then ______ is an overestimate and ______ is an underestimate.

3. Let $F(x)$ be a differentiable function. Then the linear approximation to $F(x)$ at $x = a$ is

$$F(x) \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ near } x = a.$$

Using this to estimate $F(a + \Delta x)$, we get

$$F(a+\Delta x) \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

The Accuracy of the LHS and RHS

We stated above that the LHS and the RHS both approach the same quantity (the area between the curve and the x-axis, with area below the axis considered negative) as $n \to \infty$. Let's show that they indeed do that, and along the way, get an idea of how accurate our two estimates are.

Let RHS_n and LHS_n denote the right-hand and left-hand Riemann sums respectively, both with n subintervals.

4. We're trying to show that as n approaches ∞, the RHS_n and LHS_n approach the same number.

   (a) If that's the case, what should $\lim_{n\to\infty} |RHS_n - LHS_n|$ approach?

   (b) Let's compute: $|RHS_n - LHS_n|$
   $= |$____

   $-$ ____$|$

   $= |f(a + \_\_\_\Delta x)\Delta x - f(\_\_\_)\Delta x|$
   $= |f(\_\_\_)\Delta x - f(\_\_\_)\Delta x|$
   $= |f(\_\_\_) - f(\_\_\_)|\Delta x$

   (c) Now, what happens to Δx as n approaches ∞? What does that tell you the above difference approaches as $n \to \infty$?

5. Suppose we were trying to estimate $\int_1^4 x^2 \, dx$.
   (a) To start with, let's use $n = 3$ subintervals.
      i. Compute by hand the LHS, RHS, and the difference between them.
      ii. Compute $|f(b) - f(a)|\Delta x$ in this case. Check that it matches your previous answer.
   (b) If we use $n = 6$ subintervals, what will be the difference between the LHS and the RHS?
   (c) What about $n = 60$ subintervals?
   (d) How big would you have to make n in order to get a difference of less than 0.01?

A Few More Integrals

6. (a) Evaluate $\int_{-3}^3 2x \, dx$ by drawing a picture.
   (b) Evaluate $\int_0^3 3x - 7 \, dx$ by drawing a picture.
   (c) Evaluate $\int_{-1}^1 \sqrt{1 - x^2} \, dx$ by drawing a picture. (Hint: If $y = \sqrt{1 - x^2}$, what is $x^2 + y^2$? What shape is that?)

I recommend that you read the first part of Section 5.3 about units before doing the homework for today.

So, How Does One Compute Those Areas?

Suppose $f(x)$ is continuous. We want to try to evaluate $\int_a^b f(x)\,dx$. Suppose also that $F(x)$ is an antiderivative of $f(x)$, that is, $F'(x) = f(x)$:

7. If we know the values of $F(a)$ and $F'(a) = f(a)$, we can apply linear approximation to approximate $F(a + \Delta x)$:
   - $F(a + \Delta x) \approx$ ________________ = ________________
   - Use $F'(a + \Delta x) = f(a + \Delta x)$ to approximate $F(a + 2\Delta x)$:

     $$F(a + 2\Delta x) \approx F(a + \Delta x) + \_\_\_\_\_\_\_\_$$

     $$\approx F(a) + \_\_\_\_\_\_\_\_\_\_\Delta x + \_\_\_\_\_\_\_\_\_\_\Delta x$$
   - Use $F'(a + 2\Delta x) = f(a + 2\Delta x)$ to approximate $F(a + 3\Delta x)$:

     $$F(a + 3\Delta x) \approx F(a + 2\Delta x) + \_\_\_\_\_\_\_\_$$

     $$\approx F(a) + \_\_\_\_\_\_\_\_\_\_\Delta x + \_\_\_\_\_\_\_\_\_\_\Delta x + \_\_\_\_\_\_\_\_\_\_\Delta x$$

     $$= F(a) + \sum_{i=\_\_}^{\_\_} \_\_\_\_\_\_\_\_\Delta x$$
   - Find a sum that approximates $F(a + n\Delta x)$:

     $$F(a + n\Delta x) \approx F(a) + \sum_{i=\_\_}^{\_\_} \_\_\_\_\_\_\_\_\Delta x$$

If $\Delta x = \frac{b-a}{n}$, then $a + n\Delta x =$ ___, and the sum in the last bullet point above is exactly ______ for $\int_a^b f(x)\,dx$:

$$F(\_\_\_) \approx F(a) + \_\_\_\_\_\_.$$

As the step size Δx decreases, we expect that the linear approximation gets better and better. Therefore, it makes sense that if we apply $\lim_{n\to\infty}$ so that $\Delta x \to 0$, the $\approx$ will become an $=$. We thus get the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus:
If $f(x)$ is a continuous function on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Examples

8. Use the FTC to compute the first two definite integrals from question 6. Make sure your answers match up with what you got there. (We'll do the last one in a bit.)

9. Find the area under the curve $f(x) = x^2$ between $x = 1$ and $x = 4$ exactly (Finally!). Compare your answers to the over and underestimates from question 5.

10. Compute the integral: $\int_1^2 e^{-x}\, dx$. Compare your answer to the estimates in Part II of the Riemann Sums lab.

11. Why doesn't the FTC help with computing the integral: $\int_{-1}^1 \frac{1}{t^2}\, dt$? (Hint: look at the conditions of the theorem. But also, try computing it using the FTC. What is wrong with your answer?)

12. Can you compute the integral: $\int_1^2 e^{-x^2}\, dx$? Why or why not? (Hint: You'll need an antiderivative. If you think you have found one, make sure you differentiate to check it...)

13. (a) What is $\int_{-1}^{1} \sqrt{1-x^2}\,dx$? (Refer to question 6(c))

(b) Show that $F(x) = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x$ is an antiderivative of $f(x) = \sqrt{1-x^2}$.

(Hint: show that $F'(x) = f(x)$. The algebra here is hard. If you get stuck, this isn't the most important idea here – assume that $F(x)$ is an antiderivative and move on to the next question.)

(c) Use the previous part to compute $\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

WARNING:

14. While the FTC is a very powerful theorem that allows us to compute the area under many curves, it isn't helpful in the following situations:

(a) If $f(x)$ is not ____________, the FTC does not apply (see question 11 above).

(b) If we cannot find an __________________ for $f(x)$, we can't apply the FTC in the first place (see question 12 above).

(c) Sometimes it's just easier to compute an area geometrically (see question 13 above or the questions on page 2).