Integrating to Infinity

Up to now, we've only dealt with integrals over a *finite* domain $[a, b]$. In fact, if you look back to when we proved FTC I, you'll see that it only deals with finite domains...

Definition Integrating over infinite domains (Part 1):

If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

These integrals are said to *converge* if the limits exist and are finite.

Examples

Compute the following integrals:

1. $\displaystyle\int_0^\infty e^{-2x}\,dx$

2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

3. $\displaystyle\int_{-\infty}^{-1} \frac{1}{x}\,dx$

Definition Integrating over infinite domains (Part 2):

If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx,$$

where a is any number. This integral only converges (i.e. only exists) when both integrals in the sum exist and are finite.

Example

4. Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Exercises

Calculate the following integrals, if they converge.

5. $\displaystyle\int_{0}^{\infty} \frac{e^x}{(e^x+1)^2}\,dx$ (This is a substitution. Don't forget to change bounds!)

6. $\int_0^\infty xe^{-x}\,dx$ (You may need L'Hopital's rule somewhere along the way...)

7. $\int_0^\infty \frac{1}{(x+4)^2}\,dx$

8. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2(x)\,dx$ (Hint: $\csc^2(x) = \frac{\sec^2(x)}{\tan^2(x)}$)

9. The mass of pollutants over city up to height u meters is given by $\int_0^u 25,600\pi e^{-0.0025h}\, dh$ kilograms. Compute the total mass of pollutants over the city. (Compare Varying Density Lab, Question 15.)

10. Compare the following integrals to other integrals to see if they converge.

    (a) $\int_2^\infty \frac{1}{\sqrt{x^2-1}}\, dx$
    (Hint: $\sqrt{x^2-1}$ is a little smaller than $\sqrt{x^2} = x$. What does that tell you about $\frac{1}{\sqrt{x^2-1}}$ compared to $\frac{1}{x}$?)

    (b) $\int_0^\infty \frac{1}{e^x + 2^x}\, dx$

(c) $\displaystyle\int_1^\infty \frac{1+\sin x}{x^2}\,dx$ (Hint: ______ $\leq \sin(x) \leq$ ______)

11. Find c such that $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = 1$

$$f(t) = \begin{cases} cte^{-\frac{t}{2}} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$