Review – Integrals to Infinity

1. (a) What does it mean for $\int_a^\infty f(x)\,dx$ to converge?

   (b) Does $\int_0^\infty xe^{-x^2}\,dx$ converge?

   (c) What does it mean for $\int_{-\infty}^b f(x)\,dx$ to converge?

   (d) Does $\int_{-\infty}^0 xe^{-x^2}\,dx$ converge?

   (e) What does it mean for $\int_{-\infty}^\infty f(x)\,dx$ to converge?

   (f) Does $\int_{-\infty}^\infty xe^{-x^2}\,dx$ converge?

Motivating Example

2. (a) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ using our traditional techniques.

   (b) Note that $\frac{1}{x^2} > 0$ for all x. Draw the graph of $f(x) = \frac{1}{x^2}$. What can we conclude about $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$?

   (c) What went wrong? (Also see Worksheet 6-1 Q11.)

Other Improper Integrals

3. (a) Consider $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{x}}\,dx$. Why doesn't the Fundamental Theorem immediately apply to this integral?

   (b) As done previously, we can use limits to rewrite this integral. Complete the work below. You should find that despite the fact that the function approaches ∞ as $x \to 0^+$, the area is finite! Draw the graph and shade in the area.

$$\int_{0}^{1} \frac{1}{\sqrt{x}}\,dx = \lim_{b \to 0^+} \int_{b}^{1} x^{-\frac{1}{2}}\,dx$$

4. Again, consider $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$? This time, we need to rewrite it as a sum of integrals. Complete the work below. What does your answer tell you about the area under the graph over this domain?

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

Reciprocal Power Functions

Here are the graphs of $\frac{1}{\sqrt{x}} = \frac{1}{x^{0.5}}$ and $\frac{1}{x^2}$ plotted on identical axes. Looking at the areas under the graphs between $x = 0$ and $x = 1$, we can see that $\frac{1}{x^{0.5}}$ has a thinner area as $x \to 0^+$ and $y \to \infty$ than $\frac{1}{x^2}$. As we saw above, in fact, the former is finite, and the latter infinite.

This begs the question: for what values of p is $\displaystyle\int_{0}^{1} \frac{1}{x^p}$ finite?

5. Determine whether the following integrals converge or diverge:

   (a) $\displaystyle\int_{0}^{1} \frac{1}{x}\,dx$

(b) $\displaystyle\int_0^1 \frac{1}{x^{1.01}}\,dx$

(c) $\displaystyle\int_0^1 \frac{1}{x^{0.99}}\,dx$

(d) Complete the following: $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges if ____________. That is, if ____________, the area under the graph of $\frac{1}{x^p}$ between $x = 0$ and $x = 1$ is thin enough that despite it being an infinitely 'tall' area, it is finite.

Next, we will look at the other side of these graphs: the area under them between $x = 1$ and $x = \infty$.

6. Determine whether the following integrals converge or diverge:

   (a) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

(b) $\displaystyle\int_1^\infty \frac{1}{x^{1.01}}\, dx$

(c) $\displaystyle\int_1^\infty \frac{1}{x^{0.99}}\, dx$

(d) Complete the following: $\displaystyle\int_1^\infty \frac{1}{x^p}\, dx$ converges if ____________. That is, if ____________, the area under the graph of $\frac{1}{x^p}$ between $x = 1$ and $x = \infty$ is thin enough that despite this being an infinitely 'long' area , it is finite.

7. Using your answers above, and noting that $\displaystyle\int_0^\infty f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^\infty f(x)\, dx$, determine for what value(s) of p (if any) $\displaystyle\int_0^\infty \frac{1}{x^p}\, dx$ converges.

Exercises

Calculate the following integrals:

8. $\displaystyle\int_1^e \frac{1}{x \ln x}\, dx$ (Hint: there's a substitution in there!)

9. $\displaystyle\int_0^4 \frac{8}{x^2-16}\,dx$

(Hint: you will likely need to show at some point that $\frac{1}{x-4} - \frac{1}{x+4} = \frac{8}{x^2-16}$. Do so.)