Introduction

For many years now, you have been solving *algebraic equations*, such as

$$2x + 1 = 0,\ x^2 + 3x = 1,\ e^x = 7$$

and so on. Today, we will begin to examine *differential equations*. First, let's think about what we already know:

1. (a) What does it mean to *solve* an algebraic equation? For example, what does it mean to solve $x^2 + 3x + 2 = 0$?

   (b) How do you *check* a proposed solution to an algebraic equation? For example, how do you check if $x = -2$ is a solution to $x^2 + 4 = -x^3$?

   (c) Is it easier to *solve* an algebraic equation, or to check a proposed solution? Explain.

2. (a) Is the point $x = 1$, $y = 3$ (i.e. the point $(1, 3)$) a solution to the equation $2y + x = y + 4$? Why?

   (b) Is the point $(2, 3)$ a solution to the equation $y^2 + x = y + 8$? Why?

   (c) Is the point $(1, 1)$ a solution to the equation $x^2 = 1 - y^2$? Why?

   (d) What does it mean to say that a point (a, b) is a solution to an equation?

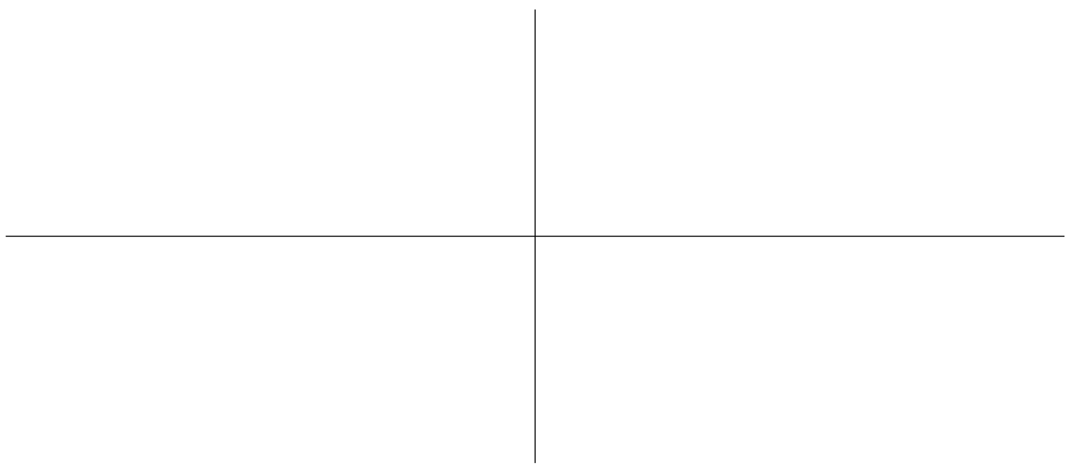
Differential Equations

Definition Equations of the form $\frac{dy}{dx} = y^2$, $\frac{dy}{dx} = 2x$ or $\frac{dy}{dx} = y + x$ are called ***differential equations***. In fact, any equation of the form $\frac{dy}{dt} = g(y, t)$ is a differential equation. $y = f(t)$ is a ***solution*** if when $f(t)$ is substituted for y in the expression $g(y, t)$, the result is $\frac{dy}{dt}$. In other words, like any other equation, when you substitute your answer into both sides of the equation, you get a true statement.

Differential Equations and Antidifferentiation

3. (a) *Check* if the function $y = x^2$ a solution to the equation $\frac{dy}{dx} = 2x$. Why?

   (b) Consider the differential equation $\frac{dy}{dx} = \cos x$. Use antidifferentiation to find a solution of this equation. Can you find more than one solution? How many can you find? Graph a few of them on the set of axes below. How are they related to each other?



In general, differential equations do not have unique solutions. In fact, a differential equation often has an infinite number of solutions, as you saw above. The reason is that we antidifferentiate, we introduce an arbitrary constant: if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any value of C. When you are asked to *solve* a differential equation, you are required to find the *general solution*. That is, the solution with a constant in it.

> **Definition** An *initial value problem* is an a differential equation with a specified value of the solution provided. Such a value is called an *initial condition*. Initial value problems most commonly have a unique solution.

4. (a) Solve the differential equation $\frac{dy}{dx} = 2e^x$.

   (b) Solve the initial value problem $\frac{dy}{dx} = 2e^x$, $y(0) = 3$.

Above, we saw differential equations of the form $\frac{dy}{dx} = f(x)$. We found that solving such an equation is just antidifferentiating $f(x)$. That is, the general solution is

$$y(x) = \int f(x)\,dx.$$

Solving differential equations is not always as straightfoward as that, though!

More Complex Differential Equations

5. Consider the differential equation $\frac{dy}{dx} = x + y$.

   (a) Why is the solution to this *not* $y(x) = \displaystyle\int x + y\,dx$? In fact, why does that integral not make any sense?

   (b) i. Check if $y(x) = \frac{x^2}{2}$ is a solution to this equation.

   ii. Check if $y(x) = e^x - x - 1$ is a solution.

   iii. Check if $y(x) = e^x - x + 1$ is a solution.

   iv. Check if $y(x) = 2e^x - x - 1$ is a solution.

   v. Is $y(x) = e^x - x + C$ a solution for any value of C? What about $y(x) = Ce^x - x - 1$?

   vi. Solve the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 4$.

6. Consider the differential equation $y'(t) = y(t)$. Complete the blank: The solution to this differential equation is a function $y(t)$ whose derivative is equal to ____________.

   (a) What function $y(t)$ satisfies the sentence you just wrote down? Plug in to both sides of the equation to check it.

   (b) Is the function $y(t) = 2 + e^t$ also a solution?

   (c) Can you figure out another function that solves the equation? Check it!

   (d) Can you write down the general solution to this equation?

7. Consider the differential equation $y'(t) = 2y(t)$. Complete the blank: The solution to this differential equation is a function $y(t)$ whose derivative is equal to ____________.

   (a) Can you find one solution to this equation?

   (b) Can you find another solution by adding a constant to your solution above?

   (c) Write down the general solution to this equation.

   (d) Solve the initial value problem $y'(t) = 2y$, $y(0) = 3$.

8. By considering the previous two questions, find the general solution of the differential equation $\frac{dy}{dx} = ky$, where k is a fixed constant. Then solve the following initial value problems:

   (a) $\frac{dy}{dt} = ky$, $y(0) = 2$.

   (b) $\frac{dy}{dt} = ky$, $y(\ln 2) = 2$.

   (c) $\frac{dy}{dt} = ky$, $y(1) = 2$.

Higher Order Differential Equations

Definition The ***order*** of a differential equation is the highest derivative that appears in it. For example $\frac{dy}{dx} = x + y$ is a *first order* equation; $\frac{d^2y}{dx^2} + \frac{dy}{dx} = y$ is a *second order* equation.

9. Consider the differential equation $\frac{d^2y}{dx^2} = -9y$.

   (a) What is the order of this equation?

   (b) Check that both $\sin(3x)$ and $\cos(3x)$ are solutions to this equation.

   (c) Check that both $2\sin(3x)$ and $3\cos(3x)$ are solutions to this equation.

(d) Check that $2\sin(3x) + 3\cos(3x)$ is a solution to this equation.

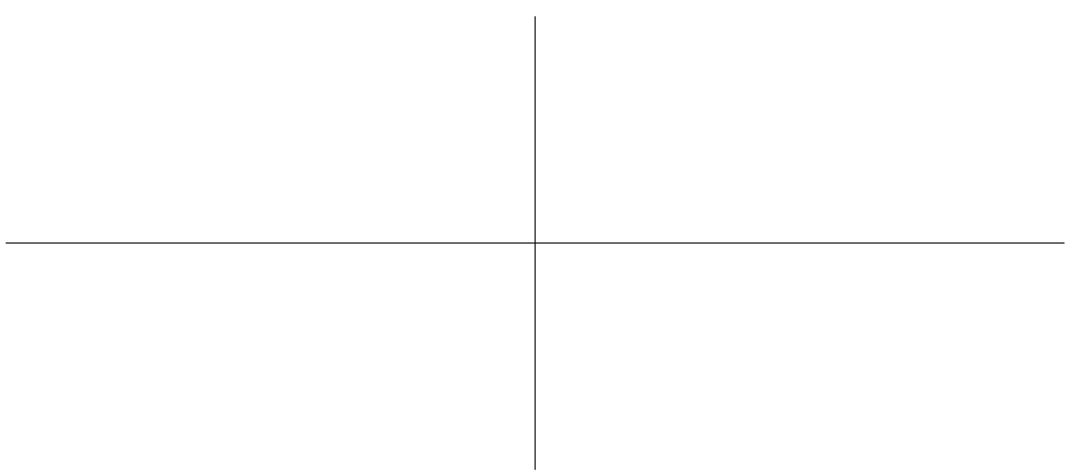
(e) Can you write down the general solution of this equation?

(f) Solve the initial value problems:

i. $\frac{d^2y}{dx^2} = -9y$, $y(0) = 1$, $y\left(\frac{\pi}{6}\right) = 0$.

ii. $\frac{d^2y}{dx^2} = -9y$, $y(0) = 1$, $y'(0) = 2$.

(g) Graph the two solutions from the previous part below.



In general, an n^{th} order differential equation has a general solution with n aribitrary constants in it.

As you can see, the solutions to a differential equation may not related to each other as simply as the ones in Question 3b above!

What we can and can't do (for now)

- The only differential equations we can currently solve analytically–that is, without (much) guessing–are those of the form

$$\frac{dy}{dt} = f(t),$$

  such as $\frac{dy}{dt} = t^2$. We solve these by finding an ____________ of $f(t)$.

- We cannot (for now) solve equations like

$$\frac{dy}{dt} = y^2.$$

  We will do (some of) these later in the semester. For now, given a solution, we can check it.

- We cannot solve second (or higher) order differential equations other than $\frac{d^2y}{dt^2} = f(t)$. We will leave other second order equations for a later course.

- We can also check whether a given function solves a particular differential equation.

Homework Questions

1. Is the function $y = -\frac{1}{x}$ a solution to the equation $\frac{dy}{dx} = y^2$? Check that $y = \frac{1}{x}$ is *not* a solution. What about $y = -\frac{1}{x+c}$? Find a solution with $y(1) = -\frac{1}{2}$.

2. Above, you found that the general solution to $\frac{dy}{dx} = ky$ is $y(x) = Ce^{kx}$. Graph solutions to the equation $\frac{dy}{dx} = 3y$ with $y(0) = 1$, $y(0) = 2$, and $y(0) = -1$ on the same set of axes.

3. What is the order of the differential equation $\frac{d^3y}{dx^3} = 24x$? Find its the general solution by antidifferentiating three times. Check that your solution has three arbitrary constants in it! Solve the initial value problems:
   (a) $\frac{d^3y}{dx^3} = 24x$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$.
   (b) $\frac{d^3y}{dx^3} = 24x$, $y(0) = 1$, $y(1) = 2$, $y(2) = 3$.

4. Solve the following initial value problems by antidifferentiation:
   (a) $\frac{dy}{dx} = x^2$, $y(1) = 1$.
   (b) $\frac{d^2h}{dt^2} = -32$, $h(0) = 100$, $h'(0) = 10$. (See the Newton's Law of Motion lab!).

5. Find a function of the form $y = x^n$ that is a solution to the differential equation $\frac{1}{2}x\frac{dy}{dx} = y$.