Review - The Exponential Model

So far we've considered growth models that look like:

$$\frac{dP}{dt} = kP,$$

for some $k > 0$. This means that the *relative growth rate* is constant:

$$\text{relative growth rate} = \frac{1}{P}\frac{dP}{dt} = k.$$

Can you think of situations where this is not a good model for growth?

Limited Resources - The Logistic Model

1. (a) When resources are limited, we expect that that relative growth rate gets larger/smaller (cross out the wrong answer) as population grows. Why?

   (b) The simplest decreasing function of P is a _____ function. This can be written as _____.

   (c) So we get $\frac{1}{P}\frac{dP}{dt} =$ _____.

   (d) For later convenience, we rearrange this as follows:

In general, when there is a limit L to how large $P(t)$ can get, a good model might be:

$$\text{relative growth rate} = \frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{L}\right).$$

2. (a) When $P \approx 0$, then relative growth rate is approximately _____

   (b) When $P \approx L$, then relative growth rate is approximately _____

This is called the **logistic growth model**.

The Logistic Model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

3. (a) Are there any equilibrium solutions?

(b) Use Geogebra to draw a slopefield for $\frac{dP}{dt}$ and classify any equilibrium solutions as stable or unstable. Sketch two solution curves: one with $P(0)$ near 0 and one with $P(0) > L$. (See the Playing with Differential Equations Lab!)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$
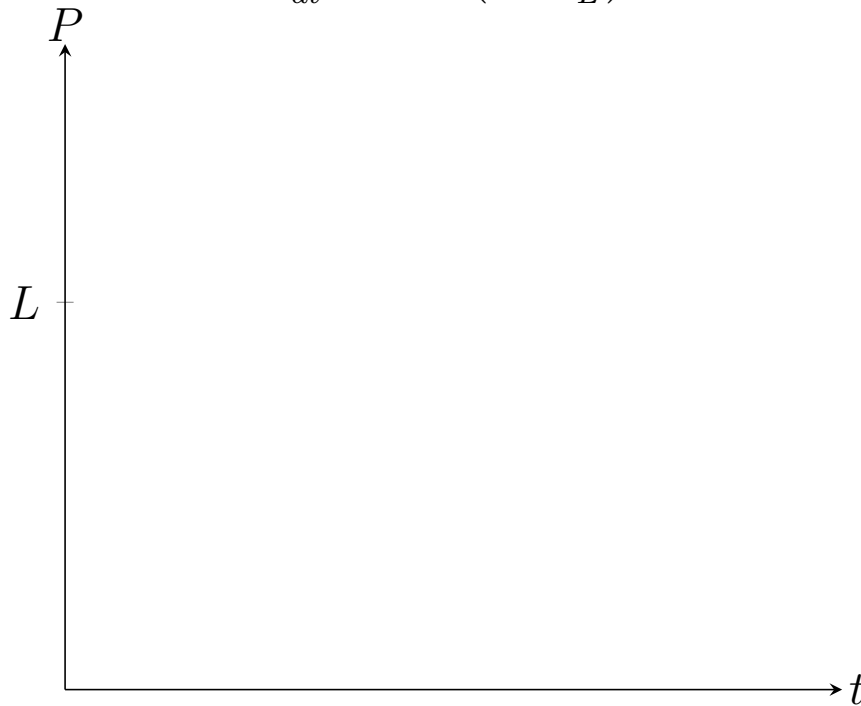
Definition   As $t \to \infty$, the population approaches L, either from above or below. We call L the *carrying capacity* of the land. It depends on things like the size of the land and availability of resources.

4. (a) Recall from 105L (and Lab 9): when a derivative has an extremum, the function has an _____. Draw a graph of $\frac{dP}{dt}$ as a function of P. For what value of P does $\frac{dP}{dt}$ reach its peak? What s is the maximal value of $\frac{dP}{dt}$?

$\frac{dP}{dt}$

$\rightarrow P$
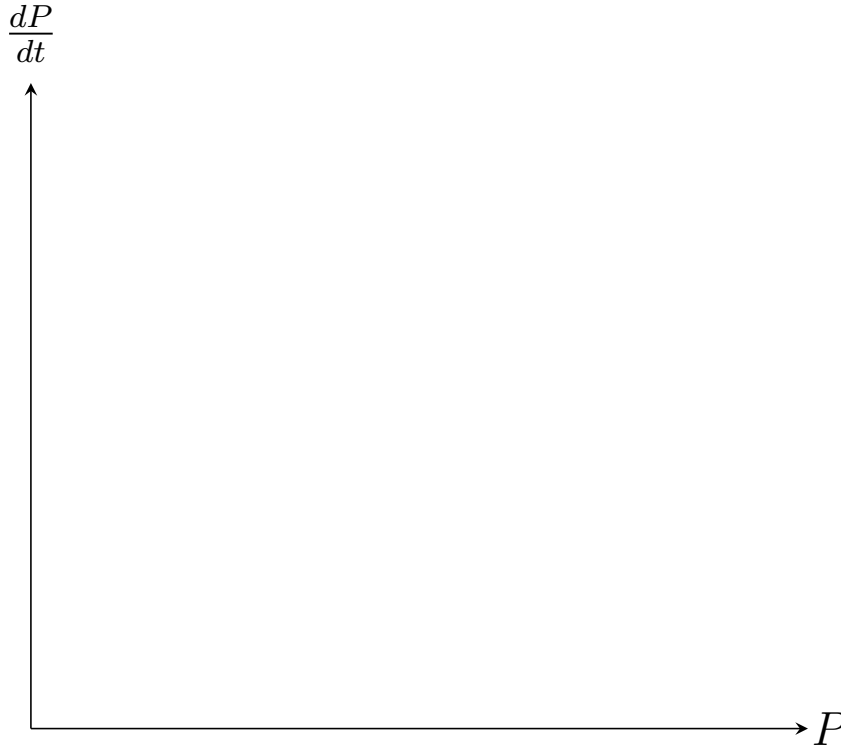
(b) Refer back to your slopefield above (from Q3), and the sketch of the solution with $P(0)$ near 0. At the value of P you found in part (a), what feature does the graph of $P(t)$ in that question have? Label this point on the graph in Q3. Also draw a tangent line to the graph in Q3 at that point and label its slope.

(c) Fill in the blank: A population that grow logistically has its maximum growth rate when the population is _____ the carrying capacity.

5. Draw two solution curves, one with $P(0) < \frac{L}{2}$, and one with $P(0) \geq \frac{L}{2}$. What do you observe regarding the concavity of your two solutions?

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$
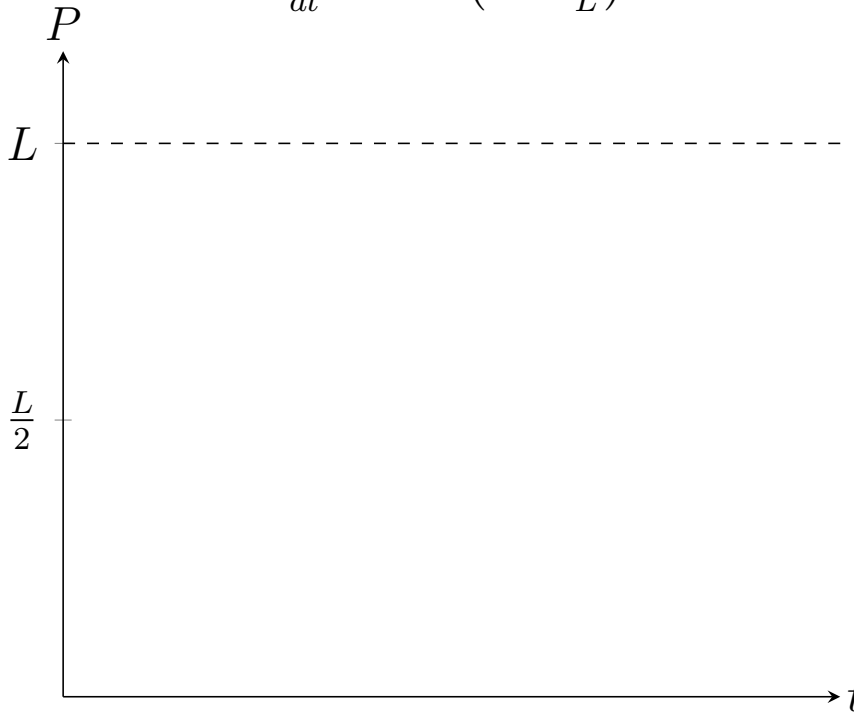
6. Assuming $P(0) = P_0$, the function $P(t) = \frac{L}{1+Ae^{-kt}}$ solves the initial value problem (you can check this by differentiating if you wish). Find A in terms of L and P_0. Show your work.

7. Assuming that $0 < P_0 < \frac{L}{2}$, how long will it take for $\frac{dP}{dt}$ to reach its maximum value? Show your work.

Summary:

8. Fill in the blanks:

   The solution to the logistic differential equation:

   $$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

   with initial condition $P(0) = P_0$ is the logistic function:

   $$P(t) = \underline{\hspace{2cm}} \qquad \text{with} \qquad A = \underline{\hspace{1.5cm}}$$

9. Fill in the blanks:

   For a logistic function,

   The maximum value of $\frac{dP}{dt}$ occurs when $P = \underline{\hspace{1.5cm}}$

   The maximal value of $\frac{dP}{dt}$ is $\underline{\hspace{1.5cm}}$.

   Time to the max rate of change $= \frac{1}{k}\ln A = \underline{\hspace{2cm}}$.

Exercise

Suppose that a population grows logistically with $k = 0.1$ years^{-1} and carrying capacity $L = 100$ thousand people. Suppose also that the initial population under consideration is 5,000 people. You may use any parts of the summary above to do the following problem.

10. Write down an initial value problem for this situation, and its solution.

11. (a) How many people are on the land when the growth rate of the population is maximal?

    (b) How long does it take for the population to reach its maximal growth rate? Include units.

    (c) What is the maximal growth rate of the population? Express your answer in people/year.

12. How long does it take for the population to reach 95% of the carrying capacity of the land? What is the growth rate (in people/yeer) at that point in time? Include units.

13. Sketch a graph of the population over time. Mark both of the points from the previous two questions on your graph. Also draw tangent lines to your graph at those points and label their slopes.